Stability of Aircraft Motion in Critical Cases

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Methods developed and/or adapted by Malkin for the analysis of the stability of equilibrium points of autonomous, nonlinear dynamic systems in certain critical cases are described. The critical cases are those in which the associated linear system obtained by an expansion about the equilibrium point has either one zero eigenvalue or a pair of pure imaginary eigenvalues, and its remaining eigenvalues have negative real parts. Application of the methods results in either a single nonlinear, autonomous, ordinary, differential equation or a pair of such equations which describe the motion in the "critical mode." Two simple examples are presented. The methods are then applied to determine the stability of the motion of a rapidly rolling aircraft in the critical case of one zero eigenvalue and of an aircraft flying at a relatively high angle of attack when the associated linear system has a pair of pure imaginary eigenvalues.

Nomenclature

$A_k(\tau), B_k(\tau)$	= coefficients of $\sin\tau$ and $\cos\tau$ in solutions for x and y
$\boldsymbol{a}$	= left-hand, real, critical eigenvector of Q; real part of left-hand, complex, critical eigenvector of Q
$\boldsymbol{b}$	= n vector formed from Nth column of Q; imaginary part of left-hand, complex, critical eigenvector of Q
b	= wing span
C_L	= lift coefficient
C_Y	= side force coefficient
C_ℓ	= rolling moment coefficient
C_m	= pitching moment coefficient
C_n	= yawing moment coefficient
$C_{\ell_j}, C_{n_j}, C_{y_j}$	= coefficients of β^j in parts of C_ℓ, C_n, C_y
$\boldsymbol{c}$	= M vector of parameters
$\bar{c}$	= mean aerodynamic chord
$\boldsymbol{c}_k$	= n vector of constants in power series expansion for $\boldsymbol{u}(x)$
$\boldsymbol{e}_j$	= complex, critical eigenvector of Q
$\boldsymbol{f}(\boldsymbol{X},\boldsymbol{c})$	= N vector of nonlinear time-derivative functions
$g(x)$	= nonlinear time derivative of x
g_k	= constant coefficient in the power series expansion of $g(x)$; constant defined by Eq. (34)
$\boldsymbol{h}(\Delta X) = \boldsymbol{Y}$	= N vector of nonlinear part of the expansion of $\boldsymbol{f}(\boldsymbol{X},\boldsymbol{c})$
I_x, I_y, I_z, I_{xz}	= moments and product of inertia of aircraft
i	= $\sqrt{-1}$
m	= aircraft mass
n	= $N-1$ if one zero critical eigenvalue; $N-2$ if a pair of imaginary eigenvalues
p,q,r	= angular velocity components
Q	= Jacobian of $\boldsymbol{f}(\boldsymbol{X},\boldsymbol{c})$ evaluated at equilibrium
S	= aircraft wing area
$\boldsymbol{u}(x)$	= particular solution for noncritical variables in zero eigenvalue case
V	= flight speed
$\boldsymbol{v}(x,y)$	= n vector solution for noncritical variables in imaginary eigenvalue case
$\Delta X = y$	= perturbation in state from equilibrium
x,y	= critical variables
$\boldsymbol{Z}$	= n vector partition of $\boldsymbol{Y}$
α	= angle of attack
β	= sideslip angle
$\Delta(\ )$	= perturbation of ( )
δ_a	= aileron deflection
δ_e	= elevator deflection
λ_j	= eigenvalue
ω	= frequency of complex critical mode
τ	= "time" variable, Eq. (28)

Subscripts

e	= equilibrium value
0	= initial value

Conventional notation is used for stability derivatives, e.g., $C_{L_\alpha} = (\partial C_L/\partial\alpha)_e$, $C_{\ell_\beta} = (\partial C_\ell/\partial\beta)_e$, $C_{m_{\dot\alpha}} = (\partial C_m/\partial\dot\alpha)\bar{c}/(2V_e)$.

Introduction

STABILITY characteristics of an equilibrium point of an autonomous, nonlinear dynamic system can usually be determined completely on the basis of the eigenvalues of the associated linear system (ALS), which is obtained via an expansion about the equilibrium point. If the eigenvalues all have negative real parts, asymptotic stability of the equilibrium point prevails; however, if at least one eigenvalue has a positive real part, the point is unstable.[1] Problems arise, however, when one or more eigenvalues of the ALS have zero, or in the practical case almost zero, real parts. These are the critical cases of the stability of motion.[2]

Critical cases are often encountered in the analysis of the stability of equilibrium states of aircraft. Some of these are considered to be of little practical concern from an analysis standpoint. For example, the critical case which occurs when the center of mass of an aircraft and its aerodynamic center coincide is well understood. However, critical cases associated with a rapid divergence of an aircraft's motion from a steady state[3,4] and states which represent the boundaries between stable motion and sustained oscillation (limit cycle)[4,5] are not as well understood. Better understanding of these is vitally important since the requirements placed on some modern military aircraft dictate that they have acceptable stability

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characteristics while flying at such large angles of attack and sideslip that linear aerodynamic models cannot be used. Furthermore, some aircraft may be required to perform predictable, essentially steady (pseudosteady) maneuvers such as rapid rolls. The existence of critical cases in the stability of the motion of high-performance aircraft has been documented.[3-5]

Analytical investigations of critical cases in the stability of the motion of aircraft, if practical, can provide much useful information concerning the equalitative aspects of motion about critical equilibrium states. Malkin[2] has developed and/or adapted several analytical methods for carrying out such investigations. His work is based on that of Lyapunov.[6] This paper concerns two of Malkin's methods, those for a single zero eigenvalue and a pair of imaginary eigenvalues, and their application to aircraft stability problems. First, each of the methods is described and illustrated through a relatively simple example. Then, each is applied to study the stability of a critical equilibrium state of an aircraft.

Methods and Examples

Malkin's methods are intimately related to more recent topological results concerning "manifolds."[4,8,9] In fact, his methods are based upon the use of a projection of the motion onto a line (plane) which is defined by the eigenvector(s) associated with the critical mode. However, the methods will be presented here without further consideration of this relationship.

In both cases treated, the dynamic system is assumed to be described by autonomous, nonlinear equations of the form,

$$\dot{X}=f(X,c) \tag{1}$$

where X is an N vector of state variables, f an N vector of functions, and c an M vector of parameters which may be interpreted as control displacements in aircraft applications. It is assumed that the elements of f are continuous functions of the elements of X and c with continuous derivatives of any subsequently desired order.

Let X_e and c_e define an equilibrium point in the space spanned by X and c, i.e.,

$$f(X_e,c_e)=0 \tag{2}$$

The stability of the equilibrium point with c fixed can be studied by expanding $f(X,c)$ about the point (X_e,c_e) to get

$$\Delta\dot{X}=Q\Delta X+h(\Delta X) \tag{3}$$

where $\Delta X=X-X_e$, Q is a constant matrix, and $h(\Delta X)$ contains only terms which are nonlinear in the elements of ΔX. If none of the eigenvalues of Q are zero, the question of stability is, as pointed out above, answered. However, if one or more of the eigenvalues of Q are zero (or, in the practical case, have very small magnitude real parts), the stability of the equilibrium point depends upon the characteristics of $h(\Delta X)$.

Single Zero Eigenvalue

Let one eigenvalue of Q be zero and let the remaining $n=N-1$ eigenvalues of Q have negative real parts. Then, for convenience, we define $y\equiv\Delta X$ and $Y\equiv h(\Delta X)$ so that Eq. (3) may be written in the form,

$$\dot{y}=Qy+Y \tag{4}$$

Let a be a left-hand eigenvector of Q corresponding to the zero eigenvalue, i.e.,

$$a^TQ=0^T \tag{5}$$

Following Malkin (Ref. 2, pp. 81-82), we define a new variable,

$$x=a^Ty \tag{6}$$

Then, by virtue of Eqs. (4) and (5),

$$\dot{x}=a^TY \tag{7}$$

Without loss of generality, we may take $a_{n+1}=1$ and solve Eq. (6) for y_{n+1}, i.e.,

$$y_{n+1}=x-a_1y_1-a_2y_2\ldots-a_ny_n \tag{8}$$

Next, we define the elements,

$$z_i=y_i \tag{9a}$$

$$b_i=q_{i,n+1} \tag{9b}$$

$$Z_i=Y_i, \qquad (i=1,2,\ldots,n) \tag{9c}$$

and

$$p_{ij}=q_{ij}-q_{i,n+1}a_j \qquad (i,j=1,2,\ldots,n) \tag{9d}$$

of the vectors z, b, Z, and the matrix P, respectively. Equation (4) can then be replaced by the n equations,

$$\dot{z}=Pz+bx+Z(z,x) \tag{10a}$$

and the single equation,

$$\dot{x}=a^TY(z,x) \tag{10b}$$

The characteristic equation of the new system is the same as that of the original system. Also, P is nonsingular.

The next step is to suppress all motion in the noncritical modes in favor of motion in the critical mode. Again following Malkin (Ref. 1, pp. 94-95), we accomplish this by finding a particular solution to Eq. (10a) of the form,

$$z=u(x) \tag{11}$$

i.e., by determining the function $u(x)$ so that

$$Pu+bx+Z[u(x)]-\mathrm{d}u/\mathrm{d}x a^TY[u(x),x] \tag{12}$$

One way to determine $u(x)$ is to set

$$u(x)=\sum_{k=1}^{m}c_kx^k \tag{13}$$

where m is an integer large enough for later purposes and the c_k are constant vectors which are such that Eq. (12) is satisfied through order m in x. Once the c_k are known, they can be used in Eq. (10b) to write

$$\dot{x}=g(x) \tag{14}$$

The problem of stability is thus reduced to determining the stability of the solutions to a single nonlinear differential equation.

The function $g(x)$ has the form,

$$g(x)=g_mx^m+g_{m+1}x^{m+1}+\ldots, \qquad m\geq 2 \tag{15}$$

Malkin has shown that the stability of the solutions to Eq. (14) is determined by g_m as follows:

1) If m is even, then the solutions are *unstable* regardless of the sign of g_m.

2) If m is odd and $g_m>0$, then the solutions are unstable. If

m is odd and $g_m < 0$, the solutions are asymptotically stable for sufficiently small initial values of x.

Application of the method, or technique, just described is probably best seen through examples. A relatively simple example follows.

Example 1: Critical Case of One Zero Eigenvalue

Consider the dynamic system,

$$\begin{bmatrix} \dot{y}_1 \\ \dot{y}_2 \end{bmatrix} = Q \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} + Y$$

where

$$Q = \begin{bmatrix} -2 & 3 \\ 2 & -3 \end{bmatrix}$$

and

$$Y = \begin{bmatrix} y_1^2 + y_2^3 \\ y_2^2 - y_1^3 \end{bmatrix}$$

The eigenvalues of Q are $\lambda_1 = 0$ and $\lambda_2 = -5$. A left-hand eigenvector of Q corresponding to $\lambda_1 = 0$ is $a = [1\ 1]^T$. It follows from Eq. (7) that the differential equation for the new variable x is

$$\dot{x} = y_1^2 + y_2^3 + y_2^2 - y_1^3$$

If we let $z_1 = y_1$, then from Eq. (8),

$$y_2 = x - z_1$$

so that

$$\dot{x} = z_1^2 + (x - z_1)^3 + (x - z_1)^2 - z_1^3 \tag{16}$$

and

$$\dot{z}_1 = -5z_1 + 3x + z_1^2 + (x - z_1)^3 \tag{17}$$

To suppress motion in the noncritical mode, we let

$$z_1 = \sum_{k=1}^{m} c_k x^k \tag{18}$$

and substitute this form of z_1 into Eq. (17), using, of course, Eq. (16). The coefficients of x and x^2 are then equated to zero to provide the equations

$$-5c_1 + 3 = 0$$

and

$$-5c_2 + c_1^2 = 0$$

Thus, $c_1 = 3/5$ and $c_2 = 9/125$. In some cases, additional coefficients may be needed, but in this case, two are sufficient.

The stability of the solution to Eq. (17) and the equilibrium point $y = 0$ may be determined from g_m. By substituting $z_1 = 3/5x + 9/125x^2$ into Eq. (17) we find

$$\dot{x} = 13/25x^2 + \ldots$$

Hence, $g_2 = 13/25$ and the origin $y = 0$ is unstable. This conclusion has been substantiated by numerically integrating the nonlinear equations.

Figure 1 shows time histories of y_1 and y_2 obtained using initial conditions $y_{10} = 0.1$ and $y_{20} = 0.1$. The motion is divergent. Interestingly, the system "jumps" from the origin to the neighboring equilibrium point (1.1089, 0.3186) which is asymptotically stable. The solution to the approximate equation for x is $x = 1/[1/x_0 - (13/25)t]$. Thus, a divergence to "infinity" by $t_\infty = 125/13$ s is predicted.

Obviously, the quadratic approximation is not valid for large x so that one cannot actually use t_∞. However, the "time to double," $t_2 = 4.81$ s, is fairly close to the actual "time to double" of about 5.5 s. Because the "time to double" depends on x_0, it is not the same as its linear counterpart. However, if an upper bound may be placed on x_0, a maximum rate of divergence can be predicted.

Pair of Imaginary Eigenvalues

The second critical case occurs when all except two of the eigenvalues of Q have negative real parts and the remaining two are pure imaginaries. For this case, we let $n = N - 2$.

We also let $e_{1,2} = a \pm ib$ be two eigenvectors of Q^T corresponding to the eigenvalues $\lambda_{1,2} = \pm i\omega$ and define the two new *scalar* variables x and y by the equations,

$$x = \tfrac{1}{2}(e_1^T + e_2^T)y = a^T y \tag{19a}$$

and

$$y = -i\tfrac{1}{2}(e_2^T - e_1^T)y = b^T y \tag{19b}$$

The differential equations for x and y are

$$\dot{x} = -\omega y + a^T Y \tag{20a}$$

and

$$\dot{y} = \omega x + b^T Y \tag{20b}$$

Next, by defining

$$z_i = y_i, \qquad (i = 1, 2, \ldots, n) \tag{21a}$$

$$G = \begin{bmatrix} a_{n+1} & a_{n+2} \\ b_{n+1} & b_{n+2} \end{bmatrix} \tag{21b}$$

and

$$K = \begin{bmatrix} a_1 & a_2 \ldots a_n \\ b_1 & b_2 \ldots b_n \end{bmatrix} \tag{21c}$$

we may rewrite Eqs. (19) as

$$\begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = Kz + G \begin{bmatrix} y_{n+1} \\ y_{n+2} \end{bmatrix} \tag{22}$$

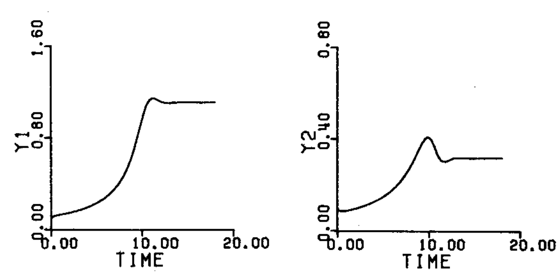

Fig. 1 State variable time histories, example 1.

Then, y_{n+1} and y_{n+2} can be exchanged for x and y by using

$$\begin{bmatrix} y_{n+1} \\ y_{n+2} \end{bmatrix} = G^{-1} \begin{bmatrix} x \\ y \end{bmatrix} - Hz \tag{23}$$

where $H = G^{-1}K$ is a $2 \times n$ matrix.

Next, we define the elements of a matrix P by

$$p_{ij} = q_{ij} - q_{i,n+1}h_{i,j} - q_{i,n+2}h_{2,j}, \qquad (i,j = 1,2,\ldots,n) \tag{24a}$$

those of a matrix B by

$$b_{ij} = q_{i,n+1}g_{1,j} + q_{i,n+2}g_{2,j}, \qquad (i = 1,2,\ldots,n, \quad j = 1,2) \tag{24b}$$

and those of a vector Z by

$$Z_i = Y_i, \qquad (i = 1,2,3,\ldots,n) \tag{24c}$$

Then, we may write

$$\dot{z} = Pz + B\begin{bmatrix} x \\ y \end{bmatrix} + Z \tag{25}$$

Motion in the noncritical modes may be suppressed by putting

$$z = v(x,y) \tag{26}$$

and choosing the vector of functions v so that Eq. (25) is satisfied, i.e.,

$$Pv + B\begin{bmatrix} x \\ y \end{bmatrix} + Z(v,x,y) - \frac{\partial v}{\partial x}[-\omega y + a^T Y] - \frac{\partial v}{\partial y}[\omega x + b^T Y] = 0 \tag{27}$$

As illustrated in a later example, the elements of $v(x,y)$ are chosen by Malkin to be power series in x and y.

Assuming for the present that the vector v is known, the stability of the solutions to Eqs. (20) with the noncritical modes suppressed can be determined in several ways (see Ref. 2, pp. 113-143), the one chosen for our work is to obtain an approximate solution to Eqs. (20) which satisfies the initial conditions $x(0) = c$, $y(0) = 0$, where c is a sufficiently small arbitrary constant (see Ref. 2, pp. 113-143).

Since the equations are nonlinear, we assume that the solution is a nonlinear oscillation. To account for the effect of nonlinearities on the "period" of motion, we introduce a new independent variable τ through the equation,

$$\tau = t\omega(1 + h_1c + h_2c^2 + \ldots)^{-1} \tag{28}$$

where $h_1, h_2, \ldots$ are arbitrary constants which may be chosen, in some cases, to eliminate secular terms from the solution. The differential equations for x and y in terms of τ are

$$\frac{dx}{d\tau} = \left(-y + \frac{1}{\omega}a^T Y\right)(1 + h_1c + h_2c^2 + \ldots) \tag{29a}$$

and

$$\frac{dy}{d\tau} = \left(x + \frac{1}{\omega}b^T Y\right)(1 + h_1c + h_2c^2 + \ldots) \tag{29b}$$

The solution to Eqs. (29) is assumed to be of the form,

$$x = \sum_{i=1}^{m} c^k x_k(\tau) \tag{30a}$$

$$y = \sum_{i=1}^{m} c^k y_k(\tau) \tag{30b}$$

Substitution of Eqs. (30) into Eqs. (29) results in the sequence of equations,

c^0:

$$dx_1/d\tau = -y_1 \qquad dy_1/d\tau = x_1 \tag{31a}$$

c^1:

$$dx_2/d\tau = -y_2 - y_1h_1 + P_2$$

$$dx_2/d\tau = x_2 + x_1h_1 + Q_2$$

$$\vdots \tag{31b}$$

where P_k and Q_k are the kth-order parts of $(1/\omega)a^TY$ and $(1/\omega)b^TY$, respectively. Assuming Y has the form of a power series in x and y, P_k and Q_k depend on only the x_j and y_j, $j<k$.

The required solutions to the first two of Eqs. (31) are $x_1 = \cos\tau$ and $y_1 = \sin\tau$, and it can be shown that

$$x_k(\tau) = A_k(\tau)\cos\tau + B_k(\tau)\sin\tau \tag{32a}$$

and

$$y_k(\tau) = A_k(\tau)\sin\tau - B_k(\tau)\cos\tau \tag{32b}$$

where

$$A_k(\tau) = \int_0^\tau [P_k(\tau)\cos\tau + Q_k(\tau)\sin\tau]\,d\tau \tag{33a}$$

and

$$B_k(\tau) = \int_0^\tau [P_k(\tau)\sin\tau - Q_k(\tau)\cos\tau - h_{k-1}]\,d\tau \tag{33b}$$

If

$$g_k = \int_0^{2\pi} [P_k(\tau)\cos\tau + Q_k(\tau)\sin\tau]\,d\tau \tag{34}$$

is zero, then $A_k(\tau)$ is periodic in τ and we can *choose* h_{k-1} so that

$$h_{k-1} = \frac{1}{2\pi}\int_0^{2\pi} [P_k(\tau)\sin\tau - Q_k(\tau)\cos\tau]\,d\tau \tag{35}$$

and $B(\tau)$ is also periodic. Hence, if $g_k = 0$ for all k, then the solution is periodic. In this event, the origin is a center, i.e., it is stable, but not asymptotically stable. If any g_k is nonzero, terms of the forms, $g_k\tau\cos\tau$ and $g_k\tau\sin\tau$, appear in the solutions for x and y. Thus, for sufficiently small c and g_m, the first nonzero g_k: if $g_m > 0$, the origin is unstable; and if $g_m < 0$, the origin is asymptotically stable.

The following example illustrates the use of this method for two imaginary eigenvalues.

Example 2: The Imaginary Eigenvalues

Consider the dynamic system of Eq. (4) with

$$Q=\begin{bmatrix} -2 & -4 & 0 \\ 1 & 0 & 1 \\ 2 & 3 & 0 \end{bmatrix}$$

and

$$Y=\begin{bmatrix} y_1^2 \\ y_2^2+2y_3^2 \\ y_1^2-y_3^2 \end{bmatrix}$$

The eigenvalues of Q are $\lambda_{1,2}=\pm i$, $\lambda_3=-2$. Left-hand eigenvectors for $\lambda_{1,2}$ are $e_{1,2}=[1 \ \pm i \ 1]^T$. Thus, $a^T=[1 \ 0 \ 1]$ and $b^T=[0 \ 1 \ 0]^T$.

From Eq. (22), with $z_1=y_1$, we have

$$\begin{bmatrix} x \\ y \end{bmatrix}=\begin{bmatrix} 1 \\ 0 \end{bmatrix} z_1+\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} y_2 \\ y_3 \end{bmatrix} \tag{36}$$

so that

$$\begin{bmatrix} y_2 \\ y_3 \end{bmatrix}=\begin{bmatrix} y \\ x \end{bmatrix}-\begin{bmatrix} 0 \\ 1 \end{bmatrix} z_1$$

The transformed equations are

$$\begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix}=\begin{bmatrix} -y \\ x \end{bmatrix}+\begin{bmatrix} z_1^2-x^2+2xz_1 \\ 2z_1^2+2x^2+y^2-4xz_1 \end{bmatrix} \tag{37a}$$

and

$$\dot{z}_1=-2z_1-4y+z_1^2 \tag{37b}$$

By putting

$$z_1=v(x,y)=v_1(x,y)+v_2(x,y)+\ldots \tag{38}$$

where v_i is a form of the ith order in x and y, the noncritical mode may be suppressed. Upon substituting Eq. (38) into Eq. (37b), we get

$$\frac{\partial v}{\partial x}(-y+v^2-x+2xv)+\frac{\partial v}{\partial y}(x+2v^2+2x^2+y^2-4xv)$$

$$=-2v-4y+v^2 \tag{39}$$

Letting $v_1=ax+by$ and $v_2=Ax^2+2Bxy+Cy^2$, we get from Eq. (39),

$$v=0.8x-1.6y-0.596x^2+0.968xy+5.636y^2$$

After some algebra, we find that, through third order in x and y,

$$\dot{x}=-y+a_1x^2+a_2xy+a_3y^2+a_4x^3+a_5x^2y+a_6xy^2+a_7y^3 \tag{40a}$$

and

$$\dot{y}=x+b_1x^2+b_2xy+b_3y^2+b_4x^3+b_5x^2y+b_6xy^2+b_7y^3 \tag{40b}$$

where $a_1=1.24$, $a_2=-5.76$, $a_3=2.56$, $a_4=-2.15$, $a_5=-5.39$, $a_6=-17.19$, $a_7=-18.04$, $b_1=0.08$, $b_2=1.28$, $b_3=6.12$, $b_4=0.477$, $b_5=3.04$, $b_6=-10.70$, and $b_7=-36.07$.

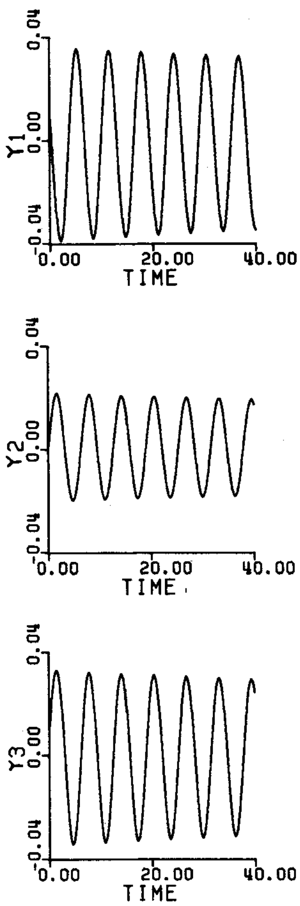

Fig. 2 State variable time histories, example 2.

A solution to Eq. (40) was sought in the form of Eqs. (30). The constant g_2 was found to be zero, but $g_3 \simeq -13.38$. Therefore, the origin $y=0$ is asymptotically stable for sufficiently small c. This conclusion has been substantiated by numerically integrating the nonlinear equations. Figure 2 shows the results. The motion in each of the three variables is a convergent oscillation.

Applications to Aircraft Problems

The stability characteristics of critical states of two aircraft (actually, *models* of aircraft) denoted A and B, were determined using the methods just described. Aircraft A, which has been the subject of a great deal of investigation,[4,7] was chosen because it is known to have "interesting" equilibrium states for which one eigenvalue is zero. Furthermore, Ross[5] showed that aircraft B has equilibrium states for which a pair of eigenvalues of the characteristic matrix have zero real

Table 1 Characteristics of aircraft A

Physical		Aerodynamic			
Mass m, kg	2718				
$\bar{c}$, m	1.829	C_{L_α}	4.35	C_{m_α}	−0.435
b, m	11.0	C_{y_β}	−0.081	C_{m_q}	−9.73
S, m	20.07	C_{ℓ_β}	−0.081	$C_{m_{\dot\alpha}}$	−2.1
V_e, m/s	152.4	C_{ℓ_p}	−0.442	$C_{m_{\delta_e}}$	−1.07
I_x kg-m^2	1254	C_{ℓ_r}	0.0309	C_{n_β}	0.0218
I_y,kg-m^2	9145	$C_{\ell_{\delta_a}}$	−0.24	C_{n_p}	0
I_z, kg-m^2	10,030			C_{n_r}	−0.0424
ρ, kg/m^3	1.2256				

Table 2 Characteristics of aircraft B

Physical		Aerodynamic			
		Linear		Nonlinear	
m, kg	2154	C_{y_p}	$0.014+0.505\alpha-0.47\alpha^2$	C_{y_1}	−0.191
b,m	3.05	C_{y_r}	0	C_{y_3}	−1.958
S, m^2	40.18	C_{ℓ_p}	−0.132	C_{ℓ_1}	−0.184
V_e, m/s	71.25	C_{ℓ_r}	$0.006+0.54\alpha$	C_{n_1}	−0.668
I_x, kg/m^2	2182	C_{n_p}	$0.0125\alpha-0.938\alpha^2$	C_{n_3}	−2.492
I_z, kg/m^2	25,430	C_{n_r}	$-0.351-0.089\alpha$		
I_{xz}, kg/m^2	1615				
ρ, kg/m^3	0.5				

Table 3 Eigenvalues and eigenvector, aircraft A

Eigenvalues	Eigenvectors ($\lambda_1=0$)
0	−0.7387
−21.882	−0.6643
−1.121	0.0209
$-2.078+4.354\,i$	0.2710
$-2.078-4.354\,i$	1

Table 4 Eigenvalues and eigenvectors, aircraft B

Eigenvalues	Eigenvectors ($\lambda_{1,2}=\pm\omega i$)
$+2.458\,i$	$-0.4018\pm0.8323\,i$
$-2.458\,i$	$0.04512\pm0.02178\,i$
−0.9109	$0.0880\pm0.03972\,i$
−0.1037	$-0.3012\pm0.2080\,i$

parts. The pertinent physical and aerodynamic data for aircraft A and B are given in Tables 1 and 2, respectively.

Aircraft A

Mehra et al.[4] considered the stability of aircraft A from the viewpoint of bifurcation theory. They used a reduced-order model predicated on the assumptions of constant aircraft speed and negligible effects of the gravitational force. Steady states (often termed "pseudosteady"[3]) in which the aircraft rolls at a constant rate are possible under these assumptions. These states may be stable or unstable[4] depending on the value of the roll rate p. For particular values of aileron deflection δ_a and elevator deflection δ_e, there exists an equilibrium point corresponding to a steady-roll state which is critical in that one eigenvalue of the associated linear system is zero.

The equations for the angle of attack α, the sideslip angle β, the roll rate p, the pitch rate q, and the yaw rate r, may be written in the forms,

$$\dot\alpha=q-p\beta-(\rho SVC_{L_\alpha}/2m)\alpha \tag{41a}$$

$$\dot\beta=p\alpha-r+(\rho SVC_{y_\beta}/2m)\beta \tag{41b}$$

$$I_x\dot p=\tfrac{1}{2}\rho SV^2b\left[C_{\ell_\beta}\beta+C_{\ell_{\delta_a}}\delta_a+\frac{b}{2V}(C_{\ell_p}p+C_{\ell_r}r)\right]-(I_z-I_y)qr \tag{41c}$$

$$I_y\dot q=\tfrac{1}{2}\rho SV^2\bar c\{C_{m_\alpha}\alpha+C_{m_{\delta_e}}\delta_e+(\bar c/2V)[(C_{m_q}+C_{m_{\dot\alpha}})q-C_{m_{\dot\alpha}}p\beta-(\rho SVC_{m_{\dot\alpha}}C_{L_\alpha}/2m)\alpha]\}-(I_x-I_z)pr \tag{41d}$$

$$I_z\dot r=\tfrac{1}{2}\rho SV^2b\left[C_{n_\beta}\beta+\frac{b}{2V}(C_{n_p}p+C_{n_r}r)\right]-(I_y-I_x)qp \tag{41e}$$

Conventional notation is used in writing Eqs. (41). That is, ρ is the atmospheric density, V the flight speed, S the wing planform area, $\bar c$ the mean aerodynamic chord, b the wing span, and I_x, I_y, and I_z the principal centroidal moments of

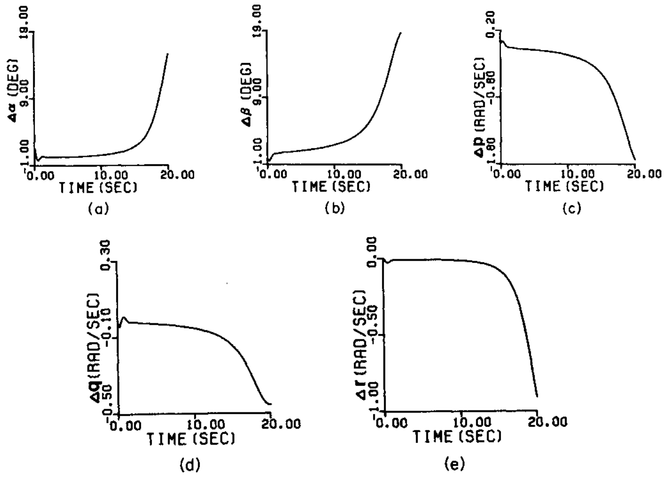

Fig. 3 Time histories of perturbations in state variables for aircraft A: a) angle of attack, b) sideslip angle, c) roll rate, d) pitch rate, e) yaw rate.

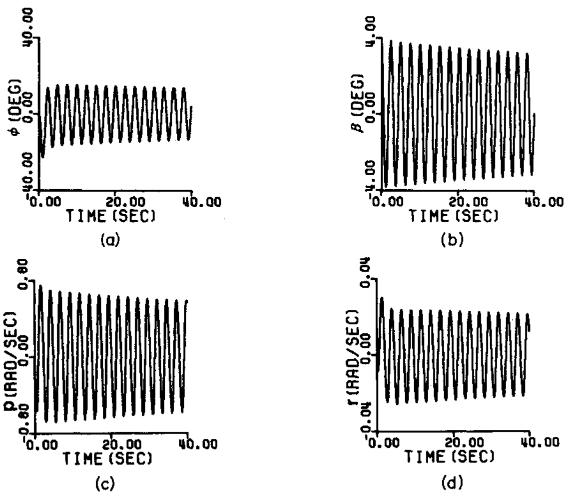

Fig. 4 Time histories of state variables for aircraft B: a) roll angle, b) sideslip angle, c) roll rate, d) yaw rate.

inertia. The usual aerodynamic derivatives have been adopted. Equations (41) can be written in the form of Eq. (4) with $y = (\Delta\alpha\Delta\beta\Delta p\Delta q\Delta r)^T$ and Q equal to the Jacobian of the right-hand sides of Eqs. (41), evaluated at an equilibrium point. Here, of course, $\Delta\alpha = \alpha - \alpha_e$, where α_e is an equilibrium value of α and the other elements have analogous meanings. Also,

$$Y = \begin{bmatrix} -\Delta p\Delta\beta \\ \Delta p\Delta\alpha \\ -[(I_z - I_y)/I_x]\Delta r\Delta q \\ -[(I_x - I_z)/I_y]\Delta p\Delta r - (\rho S V \bar{c}^2/4I_y)C_{m_{\dot\alpha}}\Delta p\Delta\beta \\ -[(I_y - I_x)/I_z]\Delta p\Delta q \end{bmatrix} \tag{42}$$

For $\delta_e = 2$ deg, the following equilibrium values were found for a critical equilibrium state for which $|Q| = 0$: $\alpha_e = -2.5627$ deg, $\beta_e = 7.8822$ deg, $p_e = -97.386$ deg/s, $q_e = 21.0825$ deg/s, and $r_e = 3.9156$ deg/s. The corresponding aileron deflection is $\delta_{a_e} = 3.817$ deg. The eigenvalues of Q are given in Table 3 along with a left-hand eigenvector corresponding to $\lambda_1 = 0$.

The method for a single zero eigenvalue was applied to this problem to get the following equation for the critical mode variable x:

$$\dot{x} = -1.0535x^2 + \ldots$$

The equilibrium point is therefore *unstable*. For $x_0 < 0$, a "time-to-double" $t_2 = 1/(2.107x_0)$ may be calculated from the truncated equation,

$$\dot{x} = -1.0535x^2$$

To test the conclusion drawn from analytical results, the full nonlinear equations were numerically integrated with all state variables (except α) equal to their equilibrium values. We set $\alpha_0 = 2$ deg, so that $\Delta\alpha_0 = 0.05727$ rad. The time histories for the perturbations in the state variables given in Fig. 3 indicate that the equilibrium point is indeed unstable. Furthermore, after the stable modes damp out (after about $t = 2$ s), the divergent time histories of $\Delta\beta$ and Δp exhibit a "time-to-double" of approximately 10 s. From the above equation for t_2, we get $t_2 \simeq 18.42$ s. The difference in the rates of divergence is attributed to the truncation of the equation for x.

Aircraft B

Ross[5] has analyzed the dynamics of aircraft B using equations of the form of Eq. (4) with $y = (\beta\phi p r)^T$, where ϕ is the roll angle of the aircraft. The elements of Q are

$$\begin{aligned}
&q_{11} = (\rho S V C_{y_1}/m), \quad q_{12} = g\cos\alpha/V, \\
&q_{13} = \sin\alpha + \rho S s C_{y_p}/m, \quad q_{14} = -\cos\alpha + \rho S s C_{y_r}/m, \\
&q_{21} = 0, \quad q_{22} = 0, \quad q_{23} = 1, \quad q_{24} = \tan\alpha, \\
&q_{31} = q_x V(I_z C_{\ell_1} + I_{xz} C_{n_1}), \quad q_{32} = 0, \\
&q_{33} = q_x s(I_z C_{\ell_p} + I_{xz} C_{n_p}), \quad q_{34} = q_x s(I_z C_{\ell_r} + I_{xz} C_{n_r}), \\
&q_{41} = q_x V(I_{xx} C_{\ell_1} + I_x C_{n_1}), \quad q_{42} = 0, \\
&q_{43} = q_x s(I_{xz} C_{\ell_p} + I_x C_{n_p}), \quad q_{44} = q_x s(I_{xz} C_{\ell_r} + I_x C_{n_r})
\end{aligned} \tag{43}$$

where $s = b/2$ and the coefficients C_{y_1}, C_{ℓ_1}, etc., are given in Table 2 in terms of the *constant* (assumed) angle of attack α. The speed is also assumed to be constant, as is the pitch angle. Moreover, for convenience, we have set $q_x = \rho S b V/(I_x I_z - I_{xz}^2)/2$. Furthermore, g is the acceleration of gravity. Finally, the nonlinear terms are

$$Y = \begin{bmatrix} (\rho b S V/m)C_{y_3}\beta^3/2 \\ 0 \\ q_x V(I_z C_{\ell_3} + I_{xz} C_{n_3})\beta^3 \\ q_x V(I_{xz} C_{\ell_3} + I_x C_{n_3})\beta^3 \end{bmatrix} \tag{44}$$

The origin, $y = 0$, is an equilibrium point and for the speed $V = 71.25$ m/s and angle of attack $\alpha = 14.3$ deg, the eigenvalues of Q are those given in Table 4. Left-hand eigenvectors for $\lambda_{1,2} = \pm 2.458i = \pm\omega i$ are also recorded in Table 4. Since only β appears in the nonlinear part of Eq. (44), the use of the methods for two imaginary roots produces the two equations,

$$\dot{x} = -\omega y + k_1\beta^3(x,y) \tag{45a}$$

and

$$\dot{y} = \omega x + k_2\beta^3(x,y) \tag{45b}$$

where $k_1 = -7.4974$, $k_2 = -5.821$, and through third order in x and y,

$$\beta(x,y) = -0.4172x + 0.969y - 0.3878x^3 + 0.04044x^2y - 0.4655xy^2 - 0.13515y^3 \tag{46}$$

An approximate solution to Eqs. (45) was sought using the techniques that have been described *supra*. It was found that $g_2 = 0$ and, by choice, $h_1 = 0$. However, $g_3 \simeq -0.777$. By referring back to the criterion for stability, we see that the origin is asymptotically stable. Figure 4 shows time histories which support this analytical results.

Conclusions

Methods attributable to Malkin and Lyapunov for the analysis of critical equilibrium states of autonomous, nonlinear dynamic systems have been described. The critical states considered are those in which either a single zero eigenvalue or a pair of imaginary eigenvalues of the ALS exist and the remaining eigenvalues all have negative real parts. Simple examples were presented to illustrate the application of the methods. The methods were then used to analyze the stability of critical equilibrium states of nonlinear mathematical models of two aircraft. The analytical results were verified by numerically integrating the nonlinear state equations.

The methods provide correct qualitative results, in the form of conclusions regarding stability. The nonlinear equations for the critical mode variables were also used to obtain estimates for the rates of divergence in the two cases of one zero eigenvalue which were considered.

Although critical equilibrium points are, in general, isolated, the nonlinear, critical-mode equations are valid within a *region* surrounding such points. Hence, they might prove useful in designing control systems which would alter the nonlinear characteristics of an aircraft without significantly altering the noncritical modes.

As with many analytical methods, the primary drawback of those described and applied herein is the amount of algebra which must be carried out to obtain the results. However, the prospects of having a *complete* stability analysis should often make the effort worthwhile.

Acknowledgment

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